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AN INDUCTIVE MODAL APPROACH FOR THE LOGIC OF EPISTEMIC INCONSISTENCY[1]

Ricardo Silvestre

Abstract
The purpose of this paper is twofold. First we want to extent a specific paranormal modal logic in such a way as obtain a paraconsistent and paracomplete multimodal logic able to formalize the notions of plausibility and certainty. With this logic at hand, and this is our second purpose, we shall use a modified version of Reiter's default logic to build a sort of inductive logic of plausibility and certainty able to represent some basic principles of epistemic inductive reasoning, such as a negative autoepistemic principle, an 'error-prone feature of induction' principle and a confirmation by enumeration principle.

Some things make the combination of modal logic and paraconsistent logic (da Costa 1974) a very interesting enterprise (Fuhrmann 1990) (da Costa and Carnielli 1986) (Goble 2006). First of all, many knowledge representation problems involving modalities seem to require a paraconsistent reasoning mechanism. An agent able to represent its beliefs and doxastic states, for example, may have evidences both to belief and not belief something; or its normative component might both require and prohibit something. Second, some have defended the idea that normal modal logic already embodies a kind of paraconsistency (Béziau 2002) (Marcos 2005) (Silvestre 2006). For instance, defining in S5 the derivated operator ~ as $\neg\Box$, we have a unary operator that does not satisfy the principle of explosion and has enough properties to be called a negation, entitling us then to classify S5 as a paraconsistent logic (Béziau 2002).

In (Silvestre 2011) and (Silvestre 2006) we presented a combination of modal and paraconsistent logic called paranormal modal logic. The motivation for this logic lies on the concept of inductive plausibility. By inductive plausibility we mean the same as Carnap's pragmatical probability (Carnap 1946), that is, a qualitative label we attach to the

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conclusion of inductive inferences. The novelty here is that when we seriously take into consideration the contradictions that are sure to arise from the use of such inferences (Perlis 1987) (Pequeno and Buchsbaum 1991), we see that there are not one, but two authentic approaches to deal with the problem (Silvestre 2007). These skeptical and credulous approaches to induction, as we have named them, give rise to two different plausibility notions which bear important relations to the field of paraconsistent and paracomplete logic (Loparíc and da Costa 1984): while the skeptical plausibility is a paracomplete notion, the credulous plausibility is a paraconsistent one. The idea of paranormal modal logic then is to analyze these two notions inside a modal framework.

First of all, we have a modal operator ? (used in a post-fixed notation) meant to represent the notion of credulous plausibility. Alike to $\Diamond$, $\alpha?$ is true iff α is true in at least one world (which we call plausible world). In addition to ?, there is the $\Box$-like operator ! meant to represent the notion of skeptical plausibility or acceptance: $\alpha!$ is true iff α is true in all plausible worlds. While the primitive negation $\neg$ is, in connection with ?, paraconsistent – we might have both $\alpha?$ and $\neg(\alpha?)$ –, in connection with ! it is paracomplete – it might be that neither $\alpha!$ nor $\neg(\alpha!)$ are true. Being its paraconsistency and paracompleteness dependent on the modality attached to the formula, we call $\neg$ a modality-dependent paranormal negation. Alike to normal modal logic, there is a family of paranormal modal logics related both axiomatically and semantically to each other. For instance, add the axiom $\alpha!\rightarrow\alpha$ ($T_?$) to $K_?$, which is the most basic paranormal modal logic, and you have the system $T_?$; add $\alpha!\rightarrow\alpha!!$ ($4_?$) to $T_?$ and you have $S4_?$; add $\alpha\rightarrow\alpha?!$ ($B_?$) to $S4_?$ and you have $S5_?$, and so on and so forth.

Considering some key aspects of the philosophical framework behind paranormal modal logic, two related combination developments can be thought. First, following the original motivation of the very first versions of paranormal modal logic (Pequeno and Buchsbaum 1991), we might think of using the notions of plausibility along with an inductive reasoning mechanism, therefore giving rise to an inductive and consequently nonmonotonic paraconsistent logic. Second, since ? and ! represent epistemic notions, it might be useful to investigate the relation between these plausibility notions and other epistemic notions. This is significant, for when we look deep at the epistemic nature of inductive inferences we see that in the same way that the conclusions of such inferences must be marked with a plausibility operator, their premises should also be referred to with the help of some epistemic notion (Silvestre 2010).

Our purpose in this paper is to advance these two combination developments. For the sake of simplicity, we shall consider only the propositional case[2]. In the next section we briefly present paranormal modal logic $K_?$. In Section 2 we introduce a multimodal logic meant to function as a logic of plausibility and certainty. In Section 3 we use this multimodal logic along with a nonmonotonic inferential mechanism to obtain a sort of logic of inductive. Finally, in the last section, we lay down some conclusive remarks.

1. Paranormal Modal Logic

As we have said, the intended meaning for the modal operators ? and ! of paranormal modal logic are the notions of credulous plausibility and skeptical plausibility or acceptability. If α is a formula, then the $\alpha?$ and $\alpha!$ mean, respectively, "α is credulously plausible" and "α is skeptically plausible or accepted". While ? is, we might say, a paraconsistent modal operator, ! is a paracomplete one: there is a model M such that both $\alpha?$ and $\neg(\alpha?)$ are satisfied in M and there is a model M such that neither $\alpha!$ nor $\neg(\alpha!)$ are satisfied in M. Both ? and ! are introduced as primitive symbols of the language. We have below the axiomatics of $K_?$, which is the most basic paranormal modal logic:

Positive Classical Axioms

P1: $\alpha\to(\beta\to\alpha)$

P2: $(\alpha\to(\beta\to\varphi))\to((\alpha\to\beta)\to(\alpha\to\varphi))$

P3: $\alpha\wedge\beta\to\alpha$

P4: $\alpha\wedge\beta\to\beta$

P5: $\alpha\to(\beta\to\alpha\wedge\beta)$

P6: $\alpha\to\alpha\vee\beta$

[2] For an account of the first order case of the class of logics to be introduced here see Silvestre (2010) and (2011).

P7: $\beta\rightarrow\alpha\vee\beta$

P8: $(\alpha\rightarrow\beta)\rightarrow((\varphi\rightarrow\beta)\rightarrow(\alpha\vee\varphi\rightarrow\beta))$

Paranormal Classical Axioms

A1: $(\alpha\rightarrow\beta)\rightarrow((\alpha\rightarrow\neg\beta)\rightarrow\neg\alpha)$, wherein β is ?-free and α is !-free

A2: $\neg\alpha\rightarrow(\alpha\rightarrow\beta)$, wherein α is ?-free

A3: $\alpha\vee\neg\alpha$, wherein α is !-free

Non-Positive Additional Classical Axioms

N1: $\neg(\alpha\rightarrow\beta)\leftrightarrow(\alpha\wedge\neg\beta)$

N2: $\neg(\alpha\wedge\beta)\leftrightarrow(\neg\alpha\vee\neg\beta)$

N3: $\neg(\alpha\vee\beta)\leftrightarrow(\neg\alpha\wedge\neg\beta)$

N4: $\neg\neg\alpha\leftrightarrow\alpha$

N5: $((\alpha\rightarrow\beta)\rightarrow\alpha)\rightarrow\alpha$

Paranormal Modal Axioms

K1: $\alpha?\leftrightarrow\sim((\sim\alpha)!)$

K2: $(\neg\alpha)!\leftrightarrow\neg(\alpha!)$

K3: $(\neg\alpha)?\leftrightarrow\neg(\alpha?)$

Modal Axioms

$K_?$: $(\alpha \rightarrow \beta)!\rightarrow(\alpha!\rightarrow\beta!)$

Rules of Inference

MP: α, $\alpha\rightarrow\beta$ / β

$N_?$: α / $\alpha!$

Axioms A1-A3 are restricted in such a way as to guarantee the paraconsistent and paracomplete behavior of ? and !, respectively. Axioms N1-N5 are there to restore the deductive power awakened by the restrictions of A1-A3. K1 sets ? and ! as the dual of each other. ~ is a derived operator meant to play the role of classical negation: $\sim\alpha =_{def} \alpha\rightarrow p\wedge\neg p$, there *p* is an arbitrary propositional symbol. Along with A1-A3, axioms K2 and K3 are the key of $K_?$'s non-classical behavior. While K2 allows us to go from the skeptical implausibility of α ($\neg(\alpha!)$) to the skeptical plausibility of $\neg\alpha$ ($(\neg\alpha)!$), K3 allows us to go from the credulous plausibility of $\neg\alpha$ ($(\neg\alpha)?$) to the credulous implausibility of α ($\neg(\alpha?)$)[3].

[3] For an explanation of the philosophical reasons behind these axioms see Silvestre (2011).

Finally, $K_?$ and $N_?$ are paranormal modal logic equivalents to axiom K and rule N, respectively.

Regarding the notion of deduction, following Fitting (Fitting 1993) we make use of the distinction between global and local premises. From a proof-theoretical point of view, the difference is that only those formulas obtained exclusively with the help of the global premises are able use the necessitation rule. In symbols we have $A ⊹ B \vdash \varphi$ as meaning that φ is deducted from A and B, A being the set of global premises and B the set of local premises. The same distinction shall be used in our definition of the notion of logical consequence.

A frame in paranormal modal logic is a pair <W,R> where W is a non-empty set of entities called worlds (or plausible worlds) and R is a binary relation on W called accessibility relation. A model then is a triple <W,R,ν> where F = <W,R> is a frame and ν is a function mapping elements of P and W to truth-values 0 and 1. We say that the model M is based on F and that $w \in W$ is a world of M. Bellow you have the semantics of $K_?$:

$\Omega_{M,w}(p) = \mho_{M,w}(p) = 1$	iff	$\nu_w(p) = 1$;
$\Omega_{M,w}(\neg\alpha) = 1$	iff	$\mho_{M,w}(\alpha) = 0$;
$\mho_{M,w}(\neg\alpha) = 1$	iff	$\Omega_{M,w}(\alpha) = 0$;
$\Omega_{M,w}(\alpha\rightarrow\beta) = 1$	iff	$\Omega_{M,w}(\alpha) = 0$ or $\Omega_{M,w}(\beta) = 1$;
$\mho_{M,w}(\alpha\rightarrow\beta) = 1$	iff	$\Omega_{M,w}(\alpha) = 0$ or $\mho_{M,w}(\beta) = 1$;
$\Omega_{M,w}(\alpha\wedge\beta) = 1$	iff	$\Omega_{M,w}(\alpha) = 1$ and $\Omega_{M,w}(\beta) = 1$;
$\mho_{M,w}(\alpha\wedge\beta) = 1$	iff	$\mho_{M,w}(\alpha) = 1$ and $\mho_{M,w}(\beta) = 1$;
$\Omega_{M,w}(\alpha\vee\beta) = 1$	iff	$\Omega_{M,w}(\alpha) = 1$ or $\Omega_{M,w}(\beta) = 1$;
$\mho_{M,w}(\alpha\vee\beta) = 1$	iff	$\mho_{M,w}(\alpha) = 1$ or $\mho_{M,w}(\beta) = 1$;
$\Omega_{M,w}(\alpha?) = 1$	iff	for some $w' \in W$ such that wRw', $\Omega_{M,w'}(\alpha) = 1$;
$\mho_{M,w}(\alpha?) = 1$	iff	for all $w' \in W$ such that wRw', $\mho_{M,w'}(\alpha) = 1$;
$\Omega_{M,w}(\alpha!) = 1$	iff	for all $w' \in W$ such that wRw', $\Omega_{M,w'}(\alpha) = 1$;
$\mho_{M,w}(\alpha!) = 1$	iff	for some $w' \in W$ such that wRw', $\mho_{M,w'}(\alpha) = 1$.

Formula α is satisfied in model M and world *w* (in symbols: $M,w\Vdash\alpha$) iff $\Omega_{M,w}(\alpha)=1$; if α is satisfied in all worlds *w* of M we say that M satisfies α (in symbols: $M\Vdash\alpha$). We then say that α is a logical consequence of A and B, A being the global premises and B the local ones (in symbols: $A \div B\vDash\varphi$) iff, given a specific set of frames F (which in $K_?$ is the set of all frames), for every model M based on F, if M satisfies all members of A, then for every world *w* of M such that $M,w\Vdash\beta$, for every $\beta\in B$, $M,w\Vdash\varphi$[4].

Ω and $\mho$ are evaluation functions which, depending on the modal operator at hand, maximize or minimize the truth-value of formulas: while Ω minimizes and $\mho$ maximizes !-marked formulas, Ω maximizes and $\mho$ minimizes ?-marked ones. As we have shown above, it is Ω which is used in the definition of the notion of satisfaction. The need of these two functions lies on the interpretation of the negation symbol $\neg$: the result of Ω applied to $\neg\alpha$ is defined in function of $\mho$, and vice-versa. This in fact is the semantic key of paranormal modal logic's non-classical behavior. $K_?$ is sound and complete (Silvestre 2011).

As we have said, exactly in the same way as it happens with normal modal logic, there is a semantic and axiomatic relation between the several paranormal modal systems. If, for instance, we restrict ourselves to the class of serial frames we obtain system $D_?$, which is syntactically obtained by adding $\alpha!\rightarrow\alpha?$ to the axiomatics of $K_?$; considering the class of all reflexive frames we have the logic $T_?$, which is syntactically obtained by adding $\alpha!\rightarrow\alpha$ to the axioms of $K_?$; taking into account the class of all reflexive and symmetric frames we obtain the system $B_?$, which is the same as $T_?$ plus axiom $\alpha\rightarrow\alpha?!$; and so on and so forth.

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[4] For more on the formalization of the notions of deduction and logical consequence inside a global-local premises framework see Fitting (1993) and Silvestre (2011).

2. A Logic of Plausibility and Certainty

What we call the logic of plausibility and certainty is a multimodal logic with two sets of operators. On the one hand we have the operators ? and ! (which as we have seen behave paraconsistently and paracompletly, respectively); on the other we have the classically behaved operators $\Box$ and $\Diamond$ meant to represent the notions of certainty and epistemological possibility: while $\Box\alpha$ means "α is certain", $\Diamond\alpha$ means "α is epistemologically possible". Alike to ! and ?, $\Box$ and $\Diamond$ are primitive symbols of the language.

An important point related to the meaning of formulas in general and non-modal formulas in particular concerns the place they appear in the relation of deductibility or logical consequence. Suppose that $A \dashv\vdash B \vDash \varphi$ (or $A \dashv\vdash B \vdash \varphi$). While an arbitrary formula α belonging to the set of global premises A can be said to mean "α is true" or "α is a true hypothesis", a formulae β belonging to the set B of local premises means simply "β is a hypothesis." This is why we can apply the N rules ($\alpha/\alpha!$ and $\alpha/\Box\alpha$) only to the global premises: since α is a true hypothesis, we sure can claim it to be skeptically plausible ($\alpha!$) as well as to be certain about its truth ($\Box\alpha$). Looking at the other way round, the fact that we can semantically conclude $\Box\alpha$ and $\alpha!$ from α (which is due to all models taken into account being exactly those in which α is true in all of its worlds) reflects the idea that α is being taken as a true hypothesis and not just as a certain or accepted one. In its turn, β helps to select, out of the multitude of worlds belonging to some of these models, the individual worlds that will be used to evaluate the conclusion φ. It therefore functions like a local, hypothetical premise whose truth is guaranteed not in all, but only in a few possible worlds of the models in question.

In addition to the axioms and inference rules of $K_?$, the logic of plausibility and certainty has the following axioms and inference rules:

Paranormal Modal Axioms

$D_?$: $\alpha! \to \alpha?$

$B_?$: $\alpha \rightarrow \alpha?!$

Normal Modal Axioms

NP: $\diamond\alpha \leftrightarrow \neg\square\neg\alpha$
K: $\square(\alpha \rightarrow \beta) \rightarrow (\square\alpha \rightarrow \square\beta)$
NN: $\sim\square\sim\alpha \leftrightarrow \neg\square\neg\alpha$
D: $\square\alpha \rightarrow \diamond\alpha$
B: $\alpha \rightarrow \square\diamond\alpha$
4: $\square\alpha \rightarrow \square\square\alpha$

Multimodal Axioms

PC: $\square\alpha \rightarrow \alpha!$

Rules of Inference

N: α / $\square\alpha$

K is system K's axiom. While NP is there to guarantee $\square$ and $\diamond$ as the dual of each other (recall that both are primitive symbols), NN is needed in order to set the normal and classical behavior of $\square$ (and, consequently, of $\diamond$.) D and $D_?$ guarantee, respectively, that what is certain is also epistemically possible and what is skeptically plausible is also credulous plausible. B and $B_?$ say, respectively, that if α is a true then it is certain that α is epistemologically possible and it is skeptically plausible that α is credulously plausible. The reasonableness of these principles is self-evident in the case where α is a true hypothesis. Concerning the local, unqualified hypothesis case, B and $B_?$ state a sort of minimal rationality principle about the hypotheses we are allowed to consider: even though they may be neither plausible nor epistemologically possible, they must be so from a second-order point of view. 4 is a sort of principle of positive introspection: if we know that α, then we know that we know that α. From B and 4 we deduce 5, $\neg\square\alpha \rightarrow \square\neg\square\alpha$, which is a principle of negative introspection: if we are not certain about α, then we are certain that we are not certain about α. PC or the plausibility-certainty axiom states that if α is certain then it is also an accepted hypothesis. From it, along with MP and K1, we obtain $\alpha? \rightarrow \diamond\alpha$, that is to say, that if α is (credulously) plausible then it is epistemically possible.

The reason why we have excluded axioms T ($\Box\alpha\rightarrow\alpha$) and $T_?$ ($\alpha!\rightarrow\alpha$) is that they represent a kind of principle of epistemic arrogance undesirable in the case of both certainty and skeptical plausibility. Taking α as meaning "α is true," while T means that if we are certain that α is true then it is true, $T_?$ means that accepting α as true entails that it is true. On similar grounds, $T_?$ and T cannot be accepted if we take α as representing an unqualified hypothesis. While from $T_?$ along with K1 we conclude $\alpha\rightarrow\alpha?$, which means that every conceivable hypothesis is automatically a plausible one, from T we derive $\alpha\rightarrow\neg\Box\neg\alpha$, which means that every conceivable hypothesis is an irrevocable one. $4_?$ ($\alpha! \rightarrow \alpha!!$) was not included on account of the desirableness of allowing gradations of credulous plausibility ($T_?$ along with K1 entails $\alpha??\rightarrow\alpha?$), from which it is possible to develop, as we shall see below, a quantitative theory of plausibility.

About the relation between our modal operators, we have that the following axioms are valid in the logic of plausibility and certainty: $\Box\alpha\rightarrow\alpha!$, that is, from certainty we obtain acceptance, $\alpha!\rightarrow\alpha?$, that is, from acceptance we obtain (credulous) plausibility, and $\alpha?\rightarrow\Diamond\alpha$, that is, from plausibility we get epistemic possibility.

A frame in the logic of plausibility and certainty is a triple $<W,R_?,R_\Diamond>$ where W is a non-empty set of worlds, $R_?$ is a binary relation on W called plausibility accessibility relation and $R_\Diamond$ is a binary relation on W called certainty accessibility relation. $R_?$ and $R_\Diamond$ satisfy the following conditions: (i) for every $w,w'\in W$ if $wR_\Diamond w'$ then $wR_?w'$, (ii) for every $w\in W$ there is at least one $w'\in W$ and at least one $w''\in W$ such that $wR_\Diamond w'$ and $wR_?w''$, (iii) for every $w,w'\in W$ if $wR_\Diamond w'$ then $w'R_\Diamond w$ and if $wR_?w'$ then $w'R_?w$, and (iv) for every $w,w',w''\in W$, if $wR_\Diamond w'$ and $w'R_\Diamond w''$ then $wR_\Diamond w''$. A *model* then is a quadruple $<W,R_?,R_\Diamond,\nu>$ where $F = <W,R_?,R_\Diamond>$ is a frame and ν is function mapping elements of P and W to truth-values 0 and 1. For the evaluation functions Ω and $\mho$ we have the following modification on what has been shown above:

$\Omega_{M,w}(\alpha?) = 1$	iff	for some $w' \in W$ such that $wR_? w'$, $\Omega_{M,w'}(\alpha) = 1$;
$\mho_{M,w}(\alpha?) = 1$	iff	for all $w' \in W$ such that $wR_? w'$, $\mho_{M,w'}(\alpha) = 1$;
$\Omega_{M,w}(\alpha!) = 1$	iff	for all $w' \in W$ such that $wR_? w'$, $\Omega_{M,w'}(\alpha) = 1$;
$\mho_{M,w}(\alpha!) = 1$	iff	for some $w' \in W$ such that $wR_? w'$, $\mho_{M,w'}(\alpha) = 1$;
$\Omega_{M,w}(\Diamond\alpha) = 1$	iff	for some $w' \in W$ such that $wR_\diamond w'$, $\Omega_{M,w'}(\alpha) = 1$;
$\mho_{M,w}(\Diamond\alpha) = 1$	iff	for some $w' \in W$ such that $wR_\diamond w'$, $\mho_{M,w'}(\alpha) = 1$;
$\Omega_{M,w}(\Box\alpha) = 1$	iff	for all $w' \in W$ such that $wR_\diamond w'$, $\Omega_{M,w'}(\alpha) = 1$;
$\mho_{M,w}(\Box\alpha) = 1$	iff	for all $w' \in W$ such that $wR_\diamond w'$, $\mho_{M,w'}(\alpha) = 1$.

The definitions of satisfatibility and logical consequence are the same as $K_?$'s. About the peculiarities of the semantics of the logic of plausibility and certainty we first note that given a frame $<W,R_\diamond,R_?>$ and a world $w \in W$, the sets $R_\diamond(w) = \{w' | wR_\diamond w'\}$ and $R_?(w) = \{w' | wR_? w'\}$ represent, respectively, what we may call the epistemically possible worlds of w and the plausible worlds of w. Second, every plausible world is also an epistemic possible world (in symbols: $R_?(w) \subseteq R_\diamond(w)$); this is restriction (i) of the frame structure, which from a proof-theoretical point of view corresponds to axiom PC. Third, all frames considered are serial frames; this is restriction (ii), which in the axiomatics corresponds to axioms D and $D_?$. Fourth, while $R_\diamond$ is a symmetric and transitive relation, $R_?$ is only a symmetric one; this, which is stated in restrictions (iii) and (iv), corresponds, respectively, to axioms B and 4 and axiom $B_?$.

3. A Logic of Inductive Implication

Traditionally the purpose of a logic of induction is one of confirmation: given a piece of evidence *e* and a hypothesis *h*, it should say whether (and possibility to what extent) *e* confirms or gives evidential support to *h* (Carnap 1950) (Hempel 1945). About the status of hypothesis *h* when *e* confirms *h* and *e* is true, despite the diversity of approaches, all theorists agree on one basic point: given that *e* confirms *h* and that *e* is true, whatever we conclude about *h* it should reflect the uncertainty inherent to inductive inferences. Almost invariably some probability notion has been chosen to do the job: even though from "*e* confirms *h*” and “*e* is true” we cannot conclude that *h* is true, we can conclude that it is probable.

This notion of probability should not be confounded with Carnap’s logical probability (Carnap 1950): while the later is supposed to be a purely logical notion connecting two sentences, the former must be seen as an epistemic label we attach to inductive conclusions in order to make explicit their defeasible character. Carnap calls this non-logical notion of probability pragmatical probability (Carnap 1946); we shall prefer the qualitative and hopefully less problematic term “inductive plausibility” or simply “plausibility”.

This characterization of induction in terms of pragmatical probability or plausibility is significant, first because considering that the truth of *e* warrants us to inductively conclude not the truth but the plausibility of *h*, we can trivially say that what *e* confirms or evidentially supports is not the truth of *h*, but its plausibility. Therefore, rather than saying that *e* confirms or inductively supports *h*, we should say that *e* confirms or inductively supports the plausibility of *h*. And given that “*h* is plausible” will possibly be inferred, the whole thing might be read as “*e* inductively implies the plausibility of *h*.” We shall call such sort of statements inductive implications.

Second, as we have mentioned, the contradictions that are sure to arise from the use inductive inferences force us to consider two different but complementary approaches to induction. A consequence of that is that sentences like “*e* confirms or evidentially supports *h*” shall necessarily mention the approach according to which the confirmation is being made. This is easily done by qualifying the plausibility notion appearing in the consequent of inductive implications: while “*e* inductively implies the credulous plausibility of *h*” characterizes a credulous approach, “*e* inductively implies the skeptical plausibility of *h*” characterizes a skeptical approach.

Third, attaching an epistemic label to the conclusions of inductive inferences leaves the door open to taking the whole notion of induction as an epistemic one. In the same way that what is confirmed or evidentially supported is not the truth of *h* but its plausibility, we may say that what confirms the plausibility of *h* is not the truth of *e*, but the certainty or plausibility of *h*[5].

As far as our formalization of these points is concerned, we shall use a version of Reiter's default logic (Reiter 1980) to represent the notion of inductive implication. The rationale behind this choice is that default logic incorporates the inferential and non-truth preserving aspects of inductive logic (Silvestre and Pequeno 2005). For example, we may quite naturally read default $\alpha{:}\varphi/\beta$ as "α inductively implies β unless $\neg\varphi$". We shall represent this by $\alpha \succsim \beta \lnsim \neg\varphi$. Second, the monotonic basis of this default logic shall be exactly the logic of certainty and plausibility just introduced in the previous section. Third, in order to capture the epistemological nature of inductive implications just mentioned, we shall force the components of our defaults to be marked with the correspondent modal operators. For instance, an inductive inference made according to a credulous approach might be represented as $\Box\alpha \succsim \beta? \lnsim \varphi$, which shall be read as "the certainty of α inductively implies the plausibility of β, unless φ".

Let $\Im$ be the language of the logic of certainty and plausibility. The inductive language $\Im_{\succ}$ built over $\Im$ is defined as follows: (i) If $\alpha \in \Im$ then $\alpha \in \Im_{\succ}$; (ii) If $\alpha,\beta,\varphi \in \Im$ then $\alpha \succsim \beta \lnsim \varphi \in \Im_{\succ}$; (iii) Nothing else belongs to $\Im_{\succ}$. We call $\alpha \succsim \beta \lnsim \varphi$ and inductive implication, being α its antecedent, β its consequent and φ its exception. $\alpha \succsim \beta$ is an abbreviation of $\alpha \succsim \beta \lnsim \bot$, $\beta \lnsim \varphi$ an abbreviation for $\top \succsim \beta \lnsim \varphi$ and β° an abbreviation for $\top \succsim \beta \lnsim \bot$, where $\bot$ is an abbreviation for $p \wedge \neg p$ and $\top$ is an abbreviation for $p \vee \neg p$, where *p* is an arbitrary propositional symbol. Any formula that is not an inductive implication is called an ordinary formula. With the help of $\Im_{\succ}$ we can define the notion of extension:

Let $A \subseteq \Im_{\succ}$ be a set of our inductive language and $S \in \Im$ a set of formulas of our multimodal language. $\Gamma(S) \subseteq \Im$ is the smallest set satisfying the following conditions: (i) $A \subseteq \Gamma(S)$; (ii) If $\Gamma(S) \div \varnothing \vdash \alpha$ then $\alpha \in \Gamma(S)$; (iii) If $\alpha \succsim \beta \lnsim \varphi \in A$, $\alpha \in \Gamma(S)$, $\varphi \notin S$ and $\sim\beta \notin S$ then

[5] For a full description of the theory of induction sketched here see (Silvestre 2007) and (Silvestre 2010).

$\beta\in\Gamma(S)$. A set of formulas E is an *extension* of A iff $\Gamma(E) = E$, that is, iff E is a fixed point of the operator Γ.

We first note that this language $\Im_{\succ}$ is a mixed language containing ordinary formulas as well as inductive implications. Therefore the set used as parameter in the definition of P-extension plays the role of both components of a default theory: it contains both a set of ordinary formulas as well as a set of inductive implications. Second, in mentioning the deduction relation of the logic of certainty and plausibility $\vdash$ we make use exclusively of global premises, the reason for that being that we want our notion of extension to incorporate the autoepistemic principle according to which we are aware of whatever our inductive mechanism infers (see below)[6]. Finally, we make the test of consistency of the consequent (in terms of ~) inside the very definition of extension, turning then $\alpha\succ\beta\not\lesssim\varphi$ into an equivalent of default $\alpha:\beta\wedge\neg\varphi/\beta$. This has the advantage of preventing so-called abnormal defaults (Morris 1988).

As one might have concluded, this inductive language does not incorporate yet the epistemological considerations we have made above about inductive inferences. As we have advanced, one way to incorporate the theory of induction we are sketching here is to require the antecedent of inductive implications to be marked with the $\Box$ symbol and the consequent with the ? symbol. We thus have what we call the epistemic inductive language $\Im_{E\succ}$: (i) If $\alpha\in\Im$ then $\alpha\in\Im_{E\succ}$; (ii) If $\alpha,\beta,\varphi\in\Im$ then $\Box\alpha\succ\beta?\not\lesssim\varphi\in\Im_{E\succ}$; (iii) Nothing else belongs to $\Im_{E\succ}$. Trivially $\Im_{E\succ}\subset\Im_{\succ}$.

In order to use this $\Im_{E\succ}$ language, we have to slightly change our definition of extension and introduce what we shall call a Δ-extension: Let $\Delta\in\Im_{\succ}-\Im$ be a set of inductive implications, $A\subseteq\Im_{E\succ}$ a set of formulas of the epistemic inductive language and $S\in\Im$ a set of formulas of our multimodal language. $\Gamma(S)\subseteq\Im$ is the smallest set satisfying the following conditions: (i) $A\subseteq\Gamma(S)$; (ii) If $\Gamma(S)\div\varnothing\vdash\alpha$ then $\alpha\in\Gamma(S)$; (iii) If

[6] To see a formulation in terms of both global and local premises see Silvestre (2010).

$\alpha \succ \beta \precnsim \varphi \in A \cup \Delta$, $\alpha \in \Gamma(S)$, $\varphi \notin S$ and $\sim\beta \notin S$ then $\beta \in \Gamma(S)$. A set of formulas E is a Δ-*extension* of A iff Γ(E) = E, that is, iff E is a fixed point of the operator Γ.

The idea here is that while A behaves like a default theory where its defaults satisfy the above motioned epistemic restrictions, Δ is a set of inductive implications meant to function like axioms able to nonmonotonically extend the inferential power of our logic of plausibility and certainty. About which inductive inferences compose Δ we have as follows.

First of all, there is the serious limitation of the logic of plausibility and certainty that we cannot conjunct plausible formulas: from α? and β? we cannot conclude (α∧β)?. The reason for that is obvious: it might be that α and β contradict each other in a strong sense ($\sim(\alpha\wedge\beta)$ or $\alpha\wedge\beta\rightarrow\bot$), in which case (α∧β)? also trivializes the theory ($\sim((\alpha\wedge\beta)?)$ or $(\alpha\wedge\beta)?\rightarrow\bot$). However, for cases where there is no contradiction between α and β it is desirable to be able to conclude (α∧β)? from α? and β?. In order to deal with that we introduce the schema of inductive implications below

$C_{?\wedge}$: $\alpha?\wedge\beta? \succ (\alpha\wedge\beta)?$

and set all instances of $C_{?\wedge}$ as belonging to Δ. See that if we have $\alpha?\wedge\beta?$ as belonging to Γ(S) and α and β contradict each other (that is to say, $\sim((\alpha\wedge\beta)?) \in S$) then (α∧β)? shall not be included in Γ(S).

Second, axiom 4, theorem 5 and rule N embody a sort of autoepistemic principle: while 4 and 5 says that we are aware of the facts we known as well as of the facts we do not know , respectively, N says that we are aware of all those propositions we take as true. But how about those statements whose truth we have no hint about? Suppose that Th(A) is all we can conclude from knowledge situation A. By N, for each $\alpha \in Th(A)$ we will have that we know that α (□α.) But how about those statements which do not belong to Th(A)? It seems reasonable that for all β such that $\beta \notin Th(A)$ we conclude ¬□β. This is what we could call a *negative autoepistemic principle*. It is trivially a nonmonotonic rule: if from A we infer ¬□β, from $A \cup \{\beta\}$ the same inference cannot be done. It therefore might formalized only with the help of an inductive implication:

NA: $((\neg\Box\alpha)?)^\circ$

NA, all instances of which belong to Δ, is the axiom which transform our system into a truly autoepistemic logic. Note that $((\neg\Box\alpha)?)^\circ$ is an abbreviation for $\top \succ (\neg\Box\alpha)? \not\precsim \bot$. Therefore, independently of the knowledge situation at hand, if it does not contain $\sim((\neg\Box\alpha)?)$ we will be able to infer nonmonotonically that $\neg\Box\alpha$ is plausible. The purpose of this is of course to make explicit that our agent does not know about the truthfulness of those formulas whose certainty cannot be inferred from his knowledge base: in the cases where $\Box\alpha$ does not belong to the logical theory, that is to say, α is not known, $(\neg\Box\alpha)?$ will be the case. One may think that because what we conclude through NA is $(\neg\Box\alpha)?$ and not $\neg\Box\alpha$, NA does not in fact perform the task we are claiming it performs. Not quite so. Since $\alpha?\rightarrow\Diamond\alpha$ (which is obtained from PP, K1 and $\Diamond\alpha\leftrightarrow\sim\Box\sim\alpha$), from $(\neg\Box\alpha)?$ we get $\Diamond\neg\Box\alpha$. From that, along with NP, we get $\neg\Box\neg\neg\Box\alpha$, which is equivalent to $\neg\Box\Box\alpha$. Since $\neg\Box\Box\alpha\rightarrow\neg\Box\alpha$, we have then that $\neg\Box\alpha$.

Third, some have defended what might be called the error-prone feature of inductive reasoning (Perlis 1987): since inductive conclusions may be mistaken even when its premises are true (something the very past use of such sort of inference has shown), any fair account of inductive reasoning should have an axiom saying that, independently of the circumstances we are working on, it is plausible that one of the beliefs we now take as rational is false. This can be formalized by the following axiom:

$I_?$: $\alpha_1?\wedge\ldots\wedge\alpha_n? \succ (\neg(\alpha_1\wedge\ldots\wedge\alpha_n))? \not\precsim (\neg(\alpha_1\wedge\ldots\wedge\alpha_n\wedge\beta))?$,

wherein $\alpha_1,\ldots,\alpha_n$ and β are *different* basic formulas

A basic formula is an atomic formula (a propositional formula) or the negation of an atomic formula. All instances of $I_?$ belong to Δ. $I_?$ says that if *n* basic formulae are plausible, then it is also plausible that some of them is false (or, as we wrote, that the negation of their conjunction is plausible.) The exception part of $I_?$ is meant to guarantee that no plausible atomic formula will be out of the conjunction $\alpha_1?\wedge\ldots\wedge\alpha_n?$: if this is the case, then the induction implication at hand cannot be used.

Finally, we have not spoken about skeptically plausible formulas. First, if we are allowed to use inductive implications only in connection to credulously plausible formulas (that is to say, inductive implications belonging to the epistemic inductive language $\mathfrak{I}_{E\succ}$), how are we to nonmonotonically introduce skeptically plausible formulas? Second, how are we to deal, in terms of inductive implications, with the relation we know there is between ?-marked formulas and !-marked ones?

One way to answer these questions is to use a very simple sort of confirmation by enumeration philosophy according to which α will be taken as accepted ($\alpha!$) only after it has got enough credulous confirmation. It is as if, by observing one black raven we turn the hypothesis "all ravens are black" into a very weakly plausible one; by observing another one we increase a little bit its degree of plausibility; and so and so forth, until that, after we have observed a certain number of black ravens, say *n*, we raise the hypothesis in question to the status of an accepted or skeptically plausible statement. In order to formalize that, we need of course to quantify how much a hypothesis was weakly confirmed or, in the context of taking weak confirmation and credulous plausibility as the same, how weakly plausible a hypothesis is.

The most straightforward way to do that is to count in how many plausible worlds a hypothesis is true. If α is true in at least one plausible world we write $\alpha?_1$; if it is true in at least two plausible worlds we write $\alpha?_2$ … until it is true in at least *n* plausible worlds, in the case we write $\alpha?_n$ or $\alpha!$. This can be done by defining the following abbreviations:

(i) $\alpha?_1 =_{def} \alpha?$;
(ii) $\alpha?_2 =_{def} (\alpha\wedge q)?\wedge(\alpha\wedge\neg q)?$, where *q* is an arbitrary atomic formula of $\mathfrak{I}$;
(i) $\alpha?_n =_{def} (\alpha\wedge p_1\wedge q)?\wedge\ldots\wedge(\alpha\wedge p_m\wedge q)?\wedge(\alpha\wedge p_1\wedge\neg q)?\wedge\ldots\wedge(\alpha\wedge p_m\wedge\neg q)?$, where $n = 2^{k+1}$, $m = 2^k$, $k>0$, $\alpha?_m \equiv (\alpha\wedge p_1)?\wedge\ldots\wedge(\alpha\wedge p_m)?$ and *q* is an arbitrary atomic formula of $\mathfrak{I}$ which do not occur in p_1;
(ii) $\alpha?_n =_{def} (\alpha\wedge p_1)?\wedge\ldots\wedge(\alpha\wedge p_n)?$, where $2^{k+1}>n>2^k$ and $\alpha?_{n+1} \equiv (\alpha\wedge p_1)?\wedge\ldots\wedge(\alpha\wedge p_n)?\wedge(\alpha\wedge p_{n+1})?$.

$\alpha?_n$ may be understood as meaning "the degree of plausibility of α is *n*." As we have mentioned above, such meaning is achieved by counting in how many plausible worlds α is true, which is performed with the help of the classical feature of worlds. Given an atomic formula *q*, we know that q and $\neg$q cannot be true at the same time in world *w*. Therefore, if $(\alpha\wedge q)?$ and $(\alpha\wedge\neg q)?$ are true, then the plausible worlds which make these two formulae true cannot be the same. Consequently, α is true in at least two worlds. Similarly, given an atomic formula *p* distinct from *q*, $(\alpha\wedge q\wedge p)?\wedge(\alpha\wedge\neg q\wedge p)?\wedge(\alpha\wedge q\wedge\neg p)?$ means that α is true in at least three worlds, $(\alpha\wedge q\wedge p)?\wedge(\alpha\wedge\neg q\wedge p)?\wedge(\alpha\wedge q\wedge\neg p)?\wedge(\alpha\wedge\neg q)?$ that α is true in at least four worlds, and so on and so forth. With the help of this abbreviation we can nonmonotonically obtain skeptically plausible formulas thought credulously plausible ones according to the confirmation by enumeration philosophy mentioned above:

$!_n$: $\alpha?_n \succ \alpha! \precsim (\neg\alpha)?$

All instances of $!_n$, for some specific *n*, belong to Δ. Note that, according to $!_n$, even if $\alpha?_n$ is true (that is, α is true in at least *n* plausible worlds) two situations might prevent $\alpha!$ from being inferred: if $\alpha!$ implies a contradiction or if $(\neg\alpha)?$ is the case. This second situation is significant, for it illustrates how the exception part can be used to set priority between inductive implications. For instance, imagine that we somehow have got $\alpha?_n$ but there is belonging to A the inductive implication $\beta\succ(\neg\alpha)?$. Suppose further that we have got β. In this case, because of the exception part of $!_n$, $\alpha!$ shall not be inferred: $\beta\succ(\neg\alpha)?$ has priority over $\alpha?_n \succ \alpha! \precsim (\neg\alpha)?$.

4. Conclusion

We have in this paper elaborated on how one might extend paranormal modal logic in such a way as to use the notions of plausibility along with an inductive reasoning mechanism which takes seriously into consideration the epistemic nature of inductive reasoning. More specifically, we introduced a non-classical multimodal logic of plausibility and certainty in which, on the one hand, the operators of plausibility ? and ! behave paraconsistently and paracompletly, respectively, and on the other hand the operators of certainty and epistemic possibility behave classically. Along with a version of Reiter's default logic, we were able to use this logic of plausibility and certainty to formalize a very simple theory of induction. It should be noted that this formalization is just one among the several possibilities we can to use the logic of plausibility and certainty along with a nonmonotonic reasoning mechanism to formalize a theory of induction. For an illustration of some of these possibilities along with the formalization of less naïve theories of induction see (Silvestre 2010).

Ricardo Silvestre
Universidade Federal de Campina Grande

ricardoss@ufcg.edu.br

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